Making sense of fractions
Part 2: fractions as a result of sharing

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In Part 1 of this article I described a double number line model of fractions which I argued might help foster students' awareness of the ratio aspect of fractions and help them develop a better sense of a fraction's size. Here, in Part 2, I consider how students might also develop their fraction sense from situations that involve division, through the activity of *fair sharing*. To do this, I draw particularly on the 3-year longitudinal study undertaken by Streefland (1991), which included children who at the start of the project were in grade 4 (ages 9-10).

The aspect of fraction that fair sharing promotes is multifaceted and different from the ratio aspect discussed in Part 1. What do we do about this complexity? Further, fair sharing can be modelled by a quite different double number line model from that used in Part 1. Is it sesible to try to develop both models early on? I return to these questions towards the end of the article.

Nunes (Nunes et al, 2006) suggests that young children may well have some "intuitive understanding of the relative nature of fractions from their experiences with division" (p1) and urges that we place a greater emphasis on division (or sharing) contexts when introducing children to fractions. Streefland (1997) puts it more bluntly: "Fractions evolve from everyday experience of fair sharing" (p347).

Students' introduction to fractions commonly involves the part-whole interpretation, typically based on the idea of partitioning a single object, such as a pizza, into equal parts. This can be thought of as a kind of sharing. The far richer sharing scenarios in studies such as Streefland's also lead to the notion of fraction as part of a whole, but the scenarios more clearly model sharing - and hence division - by having several objects to be shared rather than just one. The richness comes from the way students become engaged with the notion of equivalent fractions (through the artifice of 'table arrangements'), and from the opportunity to combine fractions by coming up with different but equivalent ways of sharing.

In Streefland's study students encountered scenarios similar to this:

> A group of 32 people go to a restaurant.
> They order 12 (extremely large) pancakes which they share equally.
> How large is each person's share?

Here we could imagine a table with 12 pancakes around which all 32 people are seated. Or, if the restaurant does not have a sufficiently large table, the group could be seated at two tables, each with 6 pancakes and 16 people, or perhaps one such table and two tables each with 3 pancakes and 8 people. Each case allows the pancakes to be shared fairly for everyone.

Figure 1 (from Streefland, 1991, p52) shows how this can become schematised through the construction of a symbol for table arrangements and by combining such symbols to form a table arrangement tree (in this case for 18 pancakes and 24 people):

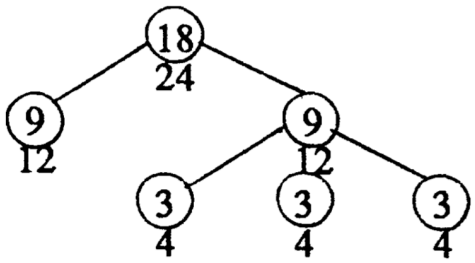


Figure 1. A table arrangement tree.

Note that the various table arrangements correspond to the equivalent fractions such as 18/24 and ¾ that show the value of a share. However, the arrangements and the fractions are not interchangeable. So, for example, while the arrangement (18)24 can be split into, say, (15)20 and (3)4, care must be taken to avoid saying $^{18}\!/_{24} = {}^{15}\!/_{20} + {}^{3}\!/_{4}$!

Returning to our example of 12 pancakes for 32 people, consider a table with just 3 pancakes and 8 people. What fraction of a pancake does each person receive? To us, it might be obvious that the answer is ⅜, but this won't necessarily be so for students. There are lots of ways in which the pancakes can be partitioned and distributed fairly. For example, two pancakes

could each be cut into 4 equal pieces, and one into 8 equal pieces, so each person receives ¼ + ⅛ of a pancake (see Figure 2a, below). Or each pancake is cut into 3 equal pieces and one of these pieces then cut into 8 equal pieces, so each person receives ⅓ + 1/24 of a pancake (see Figure2b).

As with the table arrangements, this act of sharing is highly generative, and provides a natural introduction to operating with fractions and to their equivalence. However, the complex distributions don't necessarily provide a clear link between the act of dividing 3 by 8 and the result, ⅜. Streefland resolves this by introducing the notion of 'French division', based on a fictional Alsace restaurant where the pancakes (or perhaps Flamkuchen) are delivered one at a time. In the case of a table where 3 pancakes are to be shared by 8 people, each pancake would be cut into 8 equal pieces and shared out as soon as it arrives; this would happen 3 times, so each person would eventually receive 3 lots of ⅛ of a pancake, or ⅜. Figure 2c shows this distribution, while Figure 2d shows a variant like (b) where all the pancakes are ready at the same time.

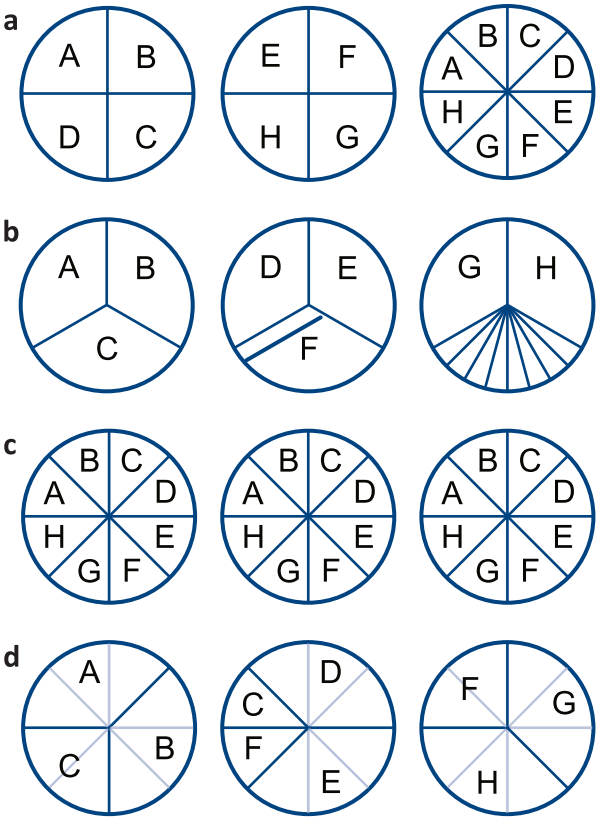


Figure 2. Sharing 3 pancakes between 8 people.

Having established that the table arrangements corresponding to the pairs of numbers 12,32 and 9,24 and 6,16 and 3,8 are equivalent, in that they result in the same size fair shares, it might still come as a surprise to students that any one of these pairs not only tells us what operation to perform to find the size of each share (12÷32 or 9÷24 or 6÷16 or 3÷8) but tells us the answer (12/32 or 9/24 or 6/16 or ⅜). So, for example, the pair 3,8 signals both

- the *operation* 3÷8 (seen as *partitive* division, whereby 3 is partitioned into 8 equal parts),
- and the *number* 3÷8, usually written as ⅜.

Nunes makes a similar point for the (rather limited) case where 1 chocolate bar is shared by 4 people: "the fraction ¼ indicates both the division - 1 divided by 4 - and the portion that each one receives" (ibid, p2).

Using the number line

Fair shares scenarios, such as 12 pancakes for 32 people, can be modelled very nicely with a double number line (DNL), but in a very different way from the ratio model from Part 1. Consider Figure 3a, where the top scale represents numbers of pancakes and the bottom scale numbers of people.

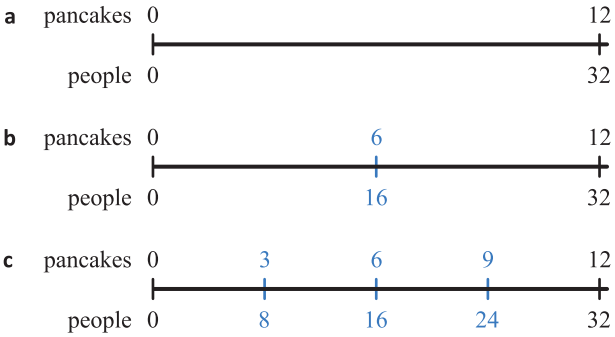


Figure 3. Sharing 12 pancakes between 32 people.

A feature of this representation is that we can use the geometry of the lines to generate equivalent arrangements of pancakes and people, that is, arrangements where the fair share that each person would receive stays the same.

In Figure 3b, the mark halfway along the line segment represents 6 pancakes shared by 16 people. Similarly in Figure 3c, the mark a quarter of the way along represents 3 pancakes and 8 people, while the mark three quarter of the way along represents 9 pancakes and 24 people.

Notice that the pairs of numbers such as 12,32, which show the table arrangements and hence

the division that can be carried out to find the value of each share, are aligned vertically on the DNL. This means that the geometry of the line reveals nothing about the size of the resulting fraction, 12⁄32 say; contrast this to the ratio model shown in Figure 4. Of course, fractions derived from pairs of numbers *along* the double number line, such as 6⁄12 = 16⁄32 = ½ still play a part in the story, but not the central part.



Figure 4. The ratio model of 12⁄32.

However, it is possible to mark on the DNL the fraction that shows a fair share, though its value and size might not be immediately obvious. This can by done by considering the arrangement where 1 person has decided to sit at a table on their own! Their portion is shown by the number *F* in Figure 5a, below.

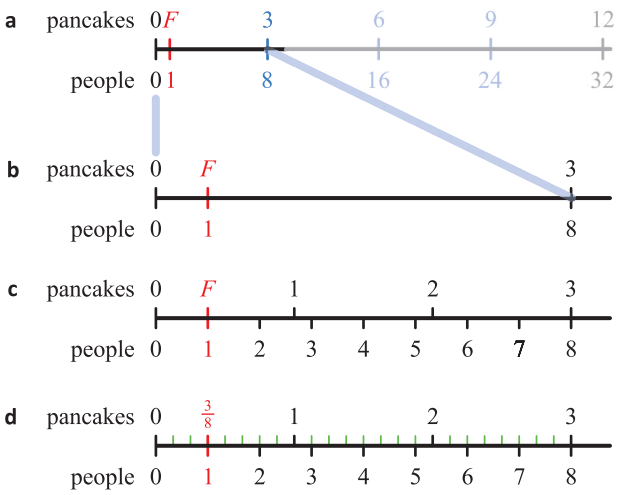


Figure 5. One person's pancakes.

We don't get much sense of the size of *F* from Figure 5a, other than that it being quite a bit less than 1. Students might get a somewhat better sense by focussing on the left hand section of the DNL shown in Figure 5b. A still better indication is given by Figure 5c, although it might still not be clear to all students that *F* is ⅜. This can be made explicit by partitioning the 3 pancakes shown on the number line into a total of 24 one-eighth parts, as in Figure 5d. However, to do this would require substantial insight.

Notice again that having started with the notion of sharing (division), we have ended up with a fraction expressing parts of a whole. We can represent the dual nature of the fraction, in this case ⅜, like this:

$$3 \div 8 = \tfrac{3}{8} \div 1 \quad \text{or} \quad \frac{3}{8} = \frac{\frac{3}{8}}{1}$$

Figure 6. 3 divided by 8 gives ⅜.

The double number line model of fair shares, where one scale represents numbers of pancakes, say, and the other numbers of people, could well help students derive the *value* of the fair share. However, as discussed above, it is not particularly well-suited to displaying the share's *size*. One could get a better sense of this by using the ratio model of fractions described in Part 1: having found the fraction ⅜, say, one could consider where 3 goes on the 0 to 8 line segment shown in Figure 7. But this would need to be done with care, and at the right time: if students try to apply the ratio model from the outset they could become thoroughly confused about where to place '3 pancakes' on a scale showing '8 people'!



Figure 7. Using a line segment to locate 3 in relation to 8.

The fair shares double number line model applies to many other multiplicative situations and at some stage one would want pupils to embrace the model as a tool to think with. However, while students are getting to grips with fair sharing, it might be worth deferring its use and letting students make do with table arrangements and table trees, which are effective models in their own right. Thus for example, a diagram like that in Figure 8 can provide an accessible way of showing how objects could be arranged and what the size of the fair shares will be.

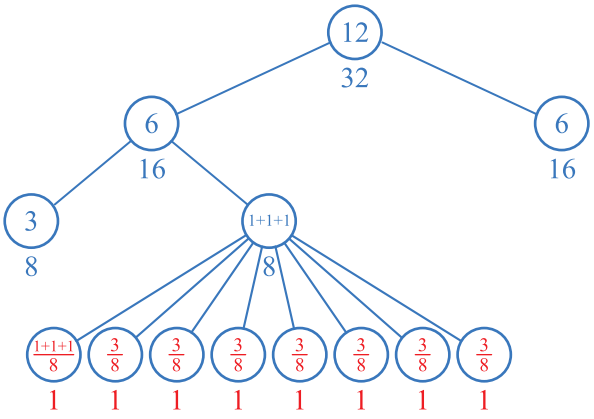


Figure 8. A table arrangement tree showing ⅜.

Interestingly, Streefland mostly uses the double number line to examine the size of *known*

fractions, rather than to model sharing. Perhaps we should do the same. His use is somewhat like that in Part 1, although he sees it as a way of developing the notion of fraction as operator rather than fraction as ratio (with ratio arising through equivalent table arrangements). He describes the use of 'mediating quantities' to compare or add fractions, whereby "The *part-whole* character of the fractions undergoes a decisive change: they become fractions in *operators*" (Streefland, 1997, p365). So, for example, ¾ of a bar of chocolate (symbolised as ¾b) could be interpreted as 9 pieces (of a bar composed of 12 pieces), or 30 cents (for a bar costing 40 cents). The corresponding (proto) double number lines are shown in Figure 9 (based on diagrams in Streefland, 1997, p365).

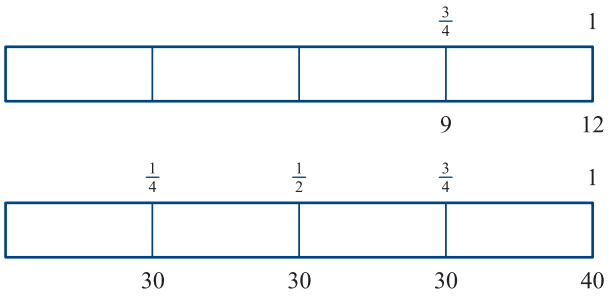


Figure 9. Using different content numbers for ¾.

Conclusion

In the fair shares context students can encounter a rich notion of fraction: as an instruction to divide, as parts of a whole resulting from division, as an operator, as something to be combined and as something that can be replaced by an equivalent. The use of the number line as described in Part 1 would add the notion of fraction as ratio and if there were a Part 3 to this article I would also propose developing the notion of fraction as a measure, as modelled, for example, by a classic wooden ruler (Figure 10).

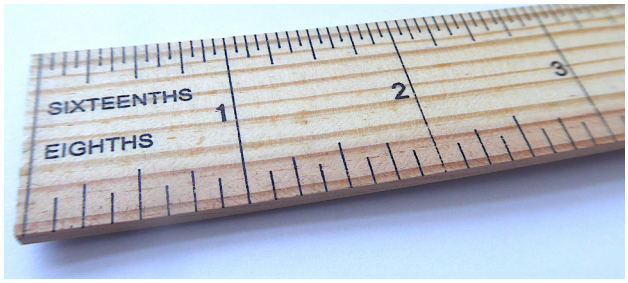


Figure 10. Fractions on a ruler.

But is it sensible to plunge students into this multifaceted world of fractions or would their learning be more secure if we initially restricted it to that of partitioning a single entity into equal parts? While I think it might be sensible to delay using the double number line to model fair sharing, I have doubts about whether narrowing students' experience of fractions would be beneficial or viable. First, students will have developed all sorts of ideas about fractions in school and from their everyday lives and it would seem dismissive of students, and frustrating for them, to pretend these don't exist. Second, if students are struggling to schematise notions of fraction involved in a familiar, accessible context they can return to the security offered by the context itself. Third, I am drawn to the Realistic Mathematics Education view (developed by Streefland and others at the Freudenthal Institute in the Netherlands) that students learn more in a rich environment where they are helped to (re)invent ideas for themselves, rather than in one where they have to try to absorb and apply ideas that have been compiled for them.

References

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Streefland, L. (1991). *Fractions in realistic mathematics education: a paradigm of developmental research*. Dordrecht: Kluwer Academic Publishers.

Streefland, L. (1997). Charming fractions or fractions being charmed? In Nunes, T., and Bryant, P. (Eds.) *Learning and Teaching Mathematics*. (pp. 347 –371). Psychology Press.